et .. $\frac{\pi a}{b} = \frac{ABab + AMa^2\frac{a}{b} + ABa^2 + AM\frac{a^2 c\mu}{b}}{Aa^2 + Bb^2 + Ma^2 - \left(1 + \frac{a}{b}\right) Aac\mu}$

et pressio integra $= \pi + \frac{\pi a}{b} =$

$$\frac{AB(a+b)^2 + AM\ (a^2 + ac\mu)\ \left(1 + \frac{a}{b}\right)}{Aa^2 + Bb^2 + Ma^2 - \left(1 + \frac{a}{b}\right) Aac\mu}$$

Si jam frictionem et pondus machinæ excludere placuerit, habetur pressio integra $= \frac{AB\,(a+b)^2}{Aa^2 + Bb^2}$: et si, ut in troclea evenit, supponatur $a = b$, erit pressio integra $= \frac{AB\,(a+a)^2}{(A+B)\,a^2} = \frac{4\,AB}{A+B}$

Christianus Hée,

*Professor Mathes. et Phys. Experim. in Statu Navali Hafniensi, Societatum Scient. Hafniensis et Berolinensis Membrum.*

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## II. *An Investigation of a General Rule for the Resolution of Isoperimetrical Problems of all Orders. By Mr.* Thomas Simpson, *F. R. S.*

Read Jan. 9, 1755.

THE different species of problems comprehended under the name of Isoperimetrical ones, are of much greater extent than the name imports; since, not only the determination of the greatest areas and solids, under equal

equal perimeters or bounds (whence the name is derived), but whatever relates to the Maxima and Minima of quantities depending on a line, ſpace, or body, whereof the figure is unknown, is, by mathematicians, included under that denomination.

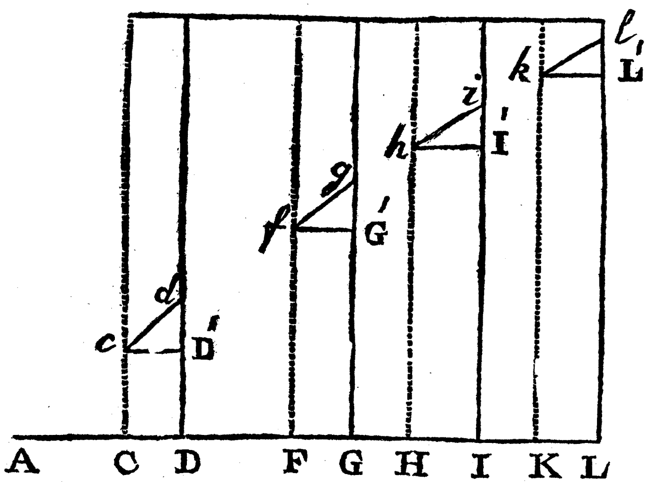
But notwithſtanding the uſefulneſs and great extent of this ſubject, nothing (that I know of) had been done thereon farther than the reſolution of certain particular caſes (ſuch as finding the line of the ſwifteſt deſcent, and the ſolid of the leaſt reſiſtance), 'till the celebrated mathematician M$^{c}$ Laurin, in his treatiſe of fluxions, gave the inveſtigation of an elegant and very eaſy method, whereby the principal problems belonging to the firſt order may be reſolved.

The paper I have now the honour to lay before the Society contains farther improvements on this ſubject: as it is by far more general than any thing yet offered, and is drawn up with a view to obviate the difficulties attending the reſolution of a very intricate kind of problems, and thereby to open an eaſy way to ſome very intereſting inquiries in natural philoſophy, I cannot doubt of its meeting with a favourable reception.

*Lemma* I.

*Fig.* 1. At any given points $D$, $G$, $I$, $L$, in a right-line $AL$, ſuppoſe perpendiculars to be erected; and from any other given points $c$, $f$, $h$, $k$, at equal diſtances ($cD'$, $fG'$, $hI'$, $kL'$,) from the ſaid perpendiculars, reſpectively, conceive right-lines $cd$, $fg$, $hi$, $kl$, to be drawn, to terminate ſomewhere in the ſaid perpendiculars; let $Q$, $R$, $S$, $T$, denote

denote any quantities expreſſed in terms of $AC$, $cD'$, and $D'd$, (independent of $Cc$) and $Q'$, $R'$, $S'$, $T'$, as many other quantities affected in the very ſame manner with $AF$, $fG'$, and $G'g$; and let $Q''$, $R''$, $S''$, $T''$, and $Q'''$, $R'''$, $S'''$, $T'''$, be quantities, ſtill, expreſſed in the ſame manner, in terms of $AH$, $hI'$, $I'i$, and $AK$, $kL'$, $L'l$, reſpectively: 'tis propoſed to find an equation expreſſing the relation of the indeterminate perpendiculars $D'd$, $G'g$, $I'i$, $L'l$, ſo that the quantity $Q+Q'+Q''+Q'''$ may be a Maximum or Minimum, at the ſame time that the values of the other quantities $R+R'+R''+R'''$, $S+S'+S''+S'''$, and $T+T'+T''+T'''$, are given, or continue invariable.




Put $D'd = \alpha$, $G'g = \beta$, $I'i = \gamma$, $L'l = \delta$; and let the fluxion of $Q$ (ſuppoſing $\alpha$ variable) be denoted by $q\dot{\alpha}$, that of $R$, by $r\dot{\alpha}$, &c. &c. then, ſince (by the nature of the propoſition) the fluxion of

$Q+$

$\mathcal{Q}+\mathcal{Q}'+\mathcal{Q}''+\mathcal{Q}'''$, as well as those of $R+R'+R''+R'''$, $S+S'+S''+S'''$, &c. muſt be equal to nothing, we therefore

have
$$\begin{cases} q\dot{\alpha}+q'\dot{\beta}+q''\dot{\gamma}+q'''\dot{\delta}=0 \\ r\dot{\alpha}+r'\dot{\beta}+r''\dot{\gamma}+r'''\dot{\delta}=0 \\ s\dot{\alpha}+s'\dot{\beta}+s''\dot{\gamma}+s'''\dot{\delta}=0 \\ t\dot{\alpha}+t'\dot{\beta}+t''\dot{\gamma}+t'''\dot{\delta}=0 \end{cases}$$

In order, now, to exterminate the fluxions $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, $\dot{\delta}$, let theſe equations be reſpectively multiplied by 1, $e$, $f$, $g$, (yet unknown), and let all the products thence ariſing be added together, whence will be had

$$\overline{q+er+fs+gt}\times\dot{\alpha}+\overline{q'+er'+fs'+gt'}\times\dot{\beta}+\overline{q''+er''+fs''+gt''}\times\dot{\gamma}+\overline{q'''+er'''+fs'''+gt'''}\times\dot{\delta}=0.$$

Make, now,
$$q'+er'+fs'+gt'=0$$
$$q''+er''+fs''+gt''=0$$
$$q'''+er'''+fs'''+gt'''=0$$

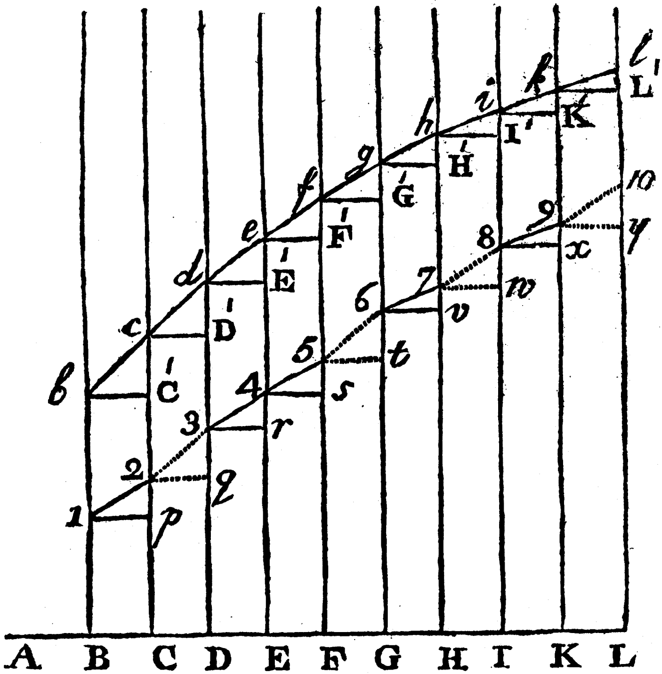
From whence (there being as many equations as quantities, $e$, $f$, $g$, to be determined), the values of theſe quantities will be always given in terms of $q'$, $r'$, $s'$, &c. that is, $e$, $f$, $g$, will always be repreſented by quantities depending on $q'$, $r'$, $s'$, &c. (or on $AF$, $G'g$, &c.) excluſive of $q$, $r$, $s$, $t$, (or of $AC$ and $D'd$), which have nothing to do in theſe laſt equations.

But, becauſe all the terms of the equation $\overline{q+er+fs+gt}\times\dot{\alpha}+\overline{q'+er'+fs'+gt'}\times\dot{\beta}$, &c. $=0$, after the firſt ($\overline{q+er+fs+gt}\times\dot{\alpha}$) do vaniſh (by their coefficients being made equal to nothing), it is evident that $q+er+fs+gt$ muſt alſo be $=0$: which is an equation expreſſing the general relation of $AC$, $cD'$, and $D'd$, with regard to to the other proposed quantities $AF$, $fG'$, $G'g$, &c. whereon the coefficients $e$, $f$, $g$, depend: and this relation will, evidently, continue the same, at whatever distances from the line $AL$, the points $c$, $f$, $h$, $k$, are taken, as these distances have nothing to do in the consideration, all the proposed quantities (as well the $Q$'s as $R$'s, &c.) being (by hypothesis) expressed in terms intirely independent thereof.

### *Lemma* II.

*Fig.* 2. Upon a given right-line $BL$, suppose perpendiculars $Bb$, $Cc$, $Dd$, &c. to be erected at equal distances; and upon the same line $BL$, as a base, suppose a polygon $Bbcdefghikl L$ to be constituted, having its angular points $b$, $c$, $d$, &c. posited in the said perpendiculars; let $y$ denote the distance of any of these perpendiculars ($Cc$, $Dd$, &c.) from any given point $A$, in $LB$ produced; and, supposing $bC'$, $cD'$, $dE'$, &c. to be drawn parallel to $AB$, let the base of any of the little triangles $bC'c$, $cD'd$, &c. be represented by $\dot{y}$, and the perpendicular corresponding by $\dot{x}$ ($\dot{y}$ being given, or the same, in every triangle, and $\dot{x}$ indeterminate): then, supposing $Q$, $R$, $S$, $T$, to denote any quantities expressed in terms of $y$, $\dot{y}$, and $\dot{x}$, it is proposed to find an equation exhibiting the general relation of the quantities $y$, $\dot{y}$, and $\dot{x}$, so that the sum of all the $\dot{y}Q$'s (resulting from the several triangles) may be a Maximum or Minimum, at the same time that the sums of all the $\dot{y}R$'s, $\dot{y}S$'s, &c. are given quantities.

Because




Becauſe the values of the quantities $\dot{y}Q$, $\dot{y}R$, $\dot{y}S$, $\dot{y}T$, depending on the different triangles $bC'c$, $cD'd$, &c. are ſuppoſed to be no-ways affected by the diſtances ($Bb$, $Cc$, &c.) of the baſes of thoſe triangles, from the baſe $BL$ of the polygon, it is evident, that thoſe values may be changed, by altering the ſpecies of one, or more, of the ſaid triangles at pleaſure, without any-ways affecting the values depending on the other triangles: for another polygon $LB$ 1 2 3 4 5, &c. may be ſo deſcribed as to have all its ſides, reſpectively, parallel to thoſe of the former, excepting only thoſe (23, 56, 78, 910) you would have to be different: ſo that the whole variation in the

 ſeveral

ſeveral ſums (whether of the $\dot{y}$ *Q*'s, $\dot{y}$ *R*'s, or $\dot{y}$ *S*'s, *&c.*) will depend intirely upon the difference of the particular triangles $2q3$, $cD'd$; $5t6$, $fG'g$, *&c.* aſſigned.

Since, therefore, the values of the $\dot{y}$ *Q*'s, $\dot{y}$ *R*'s, $\dot{y}$ *S*'s, *&c.* may be varied, at pleaſure, by altering the ſpecies of any number of correſponding triangles ($2q3$, $cD'd$; $5t6$, $fG'g$; $7w8$, $hI'i$; $9y10$, $kL'l$), while the other triangles, and the values depending on them, remain the ſame, it is manifeſt, that, when the ſum of the $\dot{y}$ *Q*'s, anſwering to all the triangles, is a Maximum or Minimum, the ſum of any number of them, taken at pleaſure (other things remaining the ſame), will likewiſe be a Maximum or Minimum; and, conſequently, that the ſum of as many *Q*'s will, at the ſame time, be a Maximum or Minimum, becauſe $\dot{y}$ is every-where the ſame, or a conſtant quantity.

Hence, if the conſtruction of the preceding Lemma be retained (ſuppoſing all the *Q*'s, *R*'s, *S*'s, *&c.* to be here expreſſed as before, in terms of $AC$, $cD'$, and $D'd$, *&c.*) it is plain that the ſum of all the *Q*'s, (or of the $\dot{y}$ *Q*'s), depending on the ſaid particular triangles (and conſequently of all the $\dot{y}$ *Q*'s in general), will be a Maximum or Minimum, when the general relation of $y$, $\dot{y}$, $\dot{x}$, (or of $AC$, $cD'$, $D'd$,) is expreſſed by the ſame equation $q+er+fs+gt=0$, there determined: in which $q$, $r$, $s$, $t$, repreſent the fluxions of *Q*, *R*, *S*, *T*, divided by that of $\dot{x}$ ($=\alpha=D'd$), and wherein the coefficients $e, f, g$, will be conſtant quantities; becauſe it is proved that their values depend intirely on the triangles $fG'g$, $hI'i$, $kL'l$, which remain the

ſame,

ſame, let the perpendicular (or ordinate) $Cc$ be taken at what diſtance you will from the given point $A$; that is, let $y$ ſtand for which you will of the diſtances $AB$, $AC$, $AD$, &c. *Q. E. I.*

*Corollary.*

If the ſides of the polygon $bcdefgh$, &c. be diminiſhed, and their number increaſed *in infinitum*, the ſum of all the $\dot{y}Q$'s will (it is well known) be expreſſed by the fluent of $\dot{y}Q$; the ſum of all the $\dot{y}R$'s, by the fluent of $\dot{y}R$, &c. whence it follows, that, to have the fluent of $\dot{y}Q$ (anſwering to a given value of $y$) a Maximum, or Minimum, and the fluents of $\dot{y}R$, $\dot{y}S$, &c. at the ſame time, given quantities, the relation of $y$, $\dot{y}$, and $\dot{x}$, muſt be defined by the equation $q + er + fs + gt = 0$, above exhibited; $q$, $r$, $s$, &c. being the reſpective fluxions of $Q$, $R$, $S$, &c. divided by that of $\dot{x}$, (or $\dot{x}$); this quantity $\dot{x}$, or $\dot{x}$, (in finding the ſaid fluxions) being, alone, conſidered as variable. Hence we have the following

GENERAL RULE.

For the reſolution of Iſoperimetrical Problems, of all orders, take the fluxions of all the given expreſſions (as well that reſpecting the Maximum, or Minimum, as of the others whoſe fluents are to be given quantities), making that quantity $(\dot{x})$ alone variable, whoſe fluent $(x)$ enters not into the ſaid expreſſions; and, having divided everywhere by the ſecond fluxion $(\ddot{x})$, let the quantities hence ariſing, joined to general coefficients, 1, $e$, $f$, $g$, &c. (whose values will depend on the values given, and may be either positive or negative), be united into one sum, and the whole be made equal to nothing; from which equation the true relation of $\dot{x}$ and $\dot{y}$, and of $x$ and $y$, will be given, let the number of restrictions be what it will.

For an example of the general Rule here laid down, let the fluxions given be $\frac{y\dot{x}^3}{\dot{y}\dot{y}}$, and $\dot{x}$; the fluent of the former, corresponding to any given value of $y$, being to be a Minimum, and that of the latter, at the same time, equal to a given quantity. Here, taking the fluxions of both expressions (making $\dot{x}$, alone, variable), and dividing by $\ddot{x}$, the quantities resulting will be $\frac{3y\dot{x}\dot{x}}{\dot{y}\dot{y}}$ and 1; so that, in this case, we have $\frac{3y\dot{x}\dot{x}}{\dot{y}\dot{y}}+e=0$, and therefore $\dot{x}=a^{\frac{1}{2}}y^{-\frac{1}{2}}\dot{y}$ (supposing $a=-\frac{1}{3}e$). From whence, by taking the fluents, $x=2a^{\frac{1}{2}}y^{\frac{1}{2}}$, or $x^2=4ay$, an equation answering to the common parabola.

If the abscisse of a curve be denoted by $x$, and the ordinate by $y$, it is known, that the several fluxions of the abscisse, curve-line, area, superficies of the generated solid, and of the solid itself, will be represented by $\dot{x}$, $\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}$, $y\dot{x}$, $2py\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}$, and $py^2\dot{x}$ respectively: if, therefore, the fluxions of these different expressions be taken as before (making $\dot{x}$, alone,

$\dot{x}$, alone, variable), we shall get $1 + \frac{e\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}$ $+ fy + \frac{gy\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}} + hy^2 = 0$, being a general equation for determining the relation of $x$ and $y$, when any one of the said five quantities (*viz.* absciſſe, curve-line, area, ſuperficies, or ſolid) is a Maximum or Minimum, and all, or any number of the others, at the ſame time, equal to given quantities; wherein the coefficients $e$, $f$, $g$, and $h$, may be either poſitive or negative, or nothing, as the caſe propoſed may require. Thus, for example, if the length of the curve, only, be given, and the area correſponding is required to be a Maximum, our equation will then become $\frac{e\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}} + fy = 0$, or $a^2\dot{x}^2 = y^2 \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}$ $\left(\text{by making } a = -\frac{e}{f}\right)$; whence $\dot{x} =$ $\frac{y\dot{y}}{\sqrt{aa-yy}}$, and conſequently $x = a - \sqrt{aa-yy}$, or $2ax - x^2 = y^2$, anſwering to a circle; which figure, therefore, of all others, contains the greateſt area, under equal bounds.

If together with the ordinate (which, here, is always ſuppoſed given) the abſciſſa, at the end of the fluent, be given likewiſe, and the ſuperficies generated by the revolution of the curve about its axis be a Minimum; then, from the ſame equation, we have $1 + \frac{gy\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}} = 0$: whence $\left(\text{making } a = -\frac{1}{g}\right)$

$\dot{x}$ is

$\dot x$ is found $= \frac{a\dot y}{\sqrt{yy-aa}}$; and, from thence, $x =$ $a \times$ hyp. log. $\frac{y+\sqrt{yy-aa}}{a}$; which equation, being impossible when $y$ is less than $a$, shews that the curve (which is here the Catanaria) cannot possibly meet the axis about which the solid is generated; and, consequently, that the case will not admit of any Minimum, unless the first, or least given value of $y$ exceeds a certain assignable magnitude.

When any, or all of the above-specified quantities are given, and the contempory fluent of some other expression, as $\overline{\dot x\dot x+\dot y\dot y}|^{n}\times y^{m}\times \dot y^{1-2n}$, is required to be a Maximum, or Minimum; the equation (by taking the fluxion of this last expression, and joining it to the former) will then be $\overline{\dot x\dot x+\dot y\dot y}|^{n-1}\times 2n\dot x y^{m}\dot y^{1-2n}$ $+d+\frac{e\dot x}{\sqrt{\dot x\dot x+\dot y\dot y}}+fy+\frac{gy\dot x}{\sqrt{\dot x\dot x+\dot y\dot y}}+hy^2=0$; which, when $m=1$, and $n=-1$, will be that defining the solid of the least resistance; and this, when the axis only is supposed to be given (without farther restrictions) will be expressed by $\overline{\dot x\dot x+\dot y\dot y}|^{-2}\times -2\dot x y\dot y^3+d=0$, or $2y\dot y^3\dot x=$ $d\times\overline{\dot x\dot x+\dot y\dot y}|^{2}$; being the case, first, considered by Sir Isaac Newton.—— If both the length and the solid content be given, the equation will be $-2\dot x y\dot y^3\times\overline{\dot x\dot x+\dot y\dot y}|^{-2}+d+hy^2=0$; but if, besides

besides these, the superficies is given likewise, it will then be $-2\dot{x}y\dot{y}^3 \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-2} + d + \frac{ey\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}$ $+ hy^2 = 0$.

Thus, in like manner, by assuming $m = -\frac{1}{2}$, and $n = \frac{1}{2}$, we have $\frac{y^{-\frac{1}{2}}\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}} + d + \frac{e\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}} +$ $fy + \frac{gy\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}} + hy^2 = 0$, for the general equation of the curve of the swiftest descent: which, when $e$ $f$, $g$, and $h$, are, all of them taken equal to nothing, will become $\frac{y^{-\frac{1}{2}}\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}} + d$; which is the case, considered by so many Others, answering to the cycloid. When the length of the arch described in the whole descent (as well as the values of $x$ and $y$) is given, the equation will then be $\frac{y^{-\frac{1}{2}}\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}$ $+ d + \frac{e\dot{x}}{\sqrt{\dot{x}\dot{x}+yy}} = 0$, or $\overline{e - y^{-\frac{1}{2}}}|^2 \times \dot{x}^2 = d^2$ $\times \overline{\dot{x}\dot{x}+y\dot{y}}$. And thus may the relation of $x$ and $y$ be determined, in other cases, and that under any number of restrictions.

III.